Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **38**, 1-17; **http://www.math.u-szeged.hu/ejqtde/**

Asymptotic stability of two dimensional systems of linear difference equations and of second order half-linear differential equations with step function coefficients

L. Hatvani[1]* and L. Székely[2]

[1]Corresponding author
University of Szeged, Bolyai Institute
Aradi vértanúk tere 1, Szeged, H-6720 Hungary
E-mail: hatvani@math.u-szeged.hu

[2]Szent István University
Institute of Mathematics and Informatics
E-mail: szekely.laszlo@gek.szie.hu

Abstract

We give a sufficient condition guaranteeing asymptotic stability with respect to x for the zero solution of the half-linear differential equation

$$x''|x'|^{n-1} + q(t)|x|^{n-1}x = 0, \qquad 1 \le n \in \mathbb{R},$$

with step function coefficient q. The geometric method of the proof can be applied also to two dimensional systems of linear non-autonomous difference equations. The application gives a new simple proof for a sharpened version of Á. Elbert's asymptotic stability theorems for such difference equations and linear second order differential equations with step function coefficients.

*Supported by the Hungarian National Foundation for Scientific Research (OTKA K75517), Analysis and Stochastics Research Group of the Hungarian Academy of Sciences, and TÁMOP 4.2.2.

Key words: asymptotic stability; Armellini-Tonelli-Sansone Theorem; step function coefficients; half-linear differential equation.

2000 Mathematics Subject Classification: Primary 34D20, 39A30.

1 Introduction

Consider the difference equation

$$\mathbf{x}_{n+1} = \mathbf{M}_n \mathbf{x}_n, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where $\mathbf{x}_n \in \mathbb{R}^2$ and $\mathbf{M}_n \in \mathbb{R}^{2\times 2}$. We do not consider the trivial case when all the entries of $\mathbf{M}_n$ are equal to 0 for some n. Let $\|\mathbf{M}\|$ be the spectral norm, i.e., $\|\mathbf{M}\|$ is the square root of the largest eigenvalue of the symmetric positive semi-definite matrix $\mathbf{M}^{\mathrm{T}}\mathbf{M}$. It is well-known [3, p. 232] that if $\prod_{n=0}^{\infty} \|\mathbf{M}_n\| = 0$, then all solutions of equation (1) tend to zero as $n \to \infty$, i.e., the zero solution is asymptotically stable. Á. Elbert [10] gave a sufficient condition for the asymptotic stability under the assumptions

(i) $\prod_{n=0}^{\infty} \max\{\|\mathbf{M}_n\|, 1\} < \infty$,

(ii) $0 < \prod_{n=0}^{\infty} \|\mathbf{M}_n\|$,

(iii) $\prod_{n=0}^{\infty} \max\{|\det \mathbf{M}_n|, 1\} < \infty$.

His proof was based on estimation of the norm of some special matrices and a "tricky" decomposition of matrices $\mathbf{M}_n$. He applied this result to deduce an Armellini-Tonelli-Sansone-type theorem (abbreviated as A-T-S theorem), i.e., a theorem guaranteeing asymptotic stability with respect to x for the zero solution of the linear second order differential equation

$$x'' + a(t)x = 0 \qquad (a(t) \nearrow \infty,\, t \to \infty) \tag{2}$$

with step function coefficient a [11, 12].

I. Bihari [5] and Elbert [9] introduced the half-linear differential equation

$$x''|x'|^{m-1} + q(t)|x|^{m-1}x = 0, \qquad m \in \mathbb{R}^+, \tag{3}$$

which has attracted attention, and it has an extensive literature (see, e.g., [7], [8] and the references therein). Bihari [6] has generalized the A-T-S theorem to this equation in the case of smooth coefficient q, requiring "regular" growth of q. Roughly speaking, this condition means that the growth of q cannot be located to a set with small measure (see Section 3). Of course, a step function q does not satisfy this condition. Elbert's method, using a wide and deep machinery from *linear* analysis, does not apply to the half-linear case.

In this paper we establish an A-T-S theorem for the half-linear differential equation with step function coefficient q. The proof is based upon a geometric method. This method applies also to the linear case, so we can give a new simple proof for Elbert's result, assuming only $\limsup_{n\to\infty}\prod_{k=0}^{n}\|\mathbf{M}_k\|<\infty$ instead of (i) – (iii).

2 Difference equation

To investigate equation (1), we will define a difference equation on the plane which has the same stability properties as equation (1). Let us introduce the following notations for the matrices of the reflection with respect to the x-axis, and of the rotation around the origin counterclockwise with φ in $\mathbb{R}^2$:

$$\mathbf{R}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \mathbf{E}(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}. \tag{4}$$

Obviously,

$$\mathbf{E}(\varphi_1)\mathbf{E}(\varphi_2)=\mathbf{E}(\varphi_1+\varphi_2),\qquad \mathbf{E}(\varphi)\mathbf{R}=\mathbf{R}\mathbf{E}(-\varphi). \tag{5}$$

We will need the following theorem (see, e.g., [16, p. 188]):

Theorem (polar factorization). *Every $\mathbf{M}\in\mathbb{R}^{n\times n}$ can be represented as a product $\mathbf{M}=\mathbf{S}\mathbf{Q}$ where $\mathbf{S}$ is symmetric, positive semi-definite, and $\mathbf{Q}$ is orthogonal. $\mathbf{S}$ is uniquely determined while $\mathbf{Q}$ is unique if and only if $\mathbf{M}$ is non-singular.*

In this theorem $\mathbf{S}$ is the square root of the symmetric positive semi-definite matrix $\mathbf{M}^{\mathrm{T}}\mathbf{M}$. If $\mathbf{M}\in\mathbb{R}^{n\times n}$ is non-singular, then the product $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ is positive definite, thus it can be diagonalized: $\mathbf{M}^{\mathrm{T}}\mathbf{M}=\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}$, where $\mathbf{D}^2$ is the diagonal matrix containing the eigenvalues of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ and the orthogonal matrix $\mathbf{P}$ has the proper eigenvectors in its columns. Then $\mathbf{S}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and

$$\mathbf{M}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{Q}. \tag{6}$$

Denote by Λ and λ the eigenvalues of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ ($\|\mathbf{M}\| = \Lambda \geq \lambda > 0$). Suppose that the diagonal elements in $\mathbf{D}$ are in decreasing order. If $\det \mathbf{M} = 0$, then $\mathbf{S}$ is positive semi-definite and the symmetric matrix $\tilde{\mathbf{S}} := \|\mathbf{M}\|^{-1}\mathbf{S}$ can be represented as $\tilde{\mathbf{S}} = \mathbf{P}\tilde{\mathbf{D}}\mathbf{P}^{-1}$, where $\mathbf{P}$ is orthogonal and

$$\tilde{\mathbf{D}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying the above argument to the coefficient matrices of (1), we have

$$\mathbf{M}_n = \|\mathbf{M}_n\|\mathbf{P}_n\hat{\mathbf{D}}_n\mathbf{P}_n^{-1}\mathbf{Q}_n, \tag{7}$$

where

$$\hat{\mathbf{D}}_n := \begin{pmatrix} 1 & 0 \\ 0 & d_n \end{pmatrix}, \qquad d_n := \begin{cases} \sqrt{\frac{\lambda_n}{\Lambda_n}} > 0, & \text{if } \det \mathbf{M}_n \neq 0; \\ 0, & \text{if } \det \mathbf{M}_n = 0. \end{cases} \tag{8}$$

Let us examine the flow $\mathbf{F}_n := \prod_{k=0}^{n} \mathbf{M}_k$ of equation (1). Using the fact, that the product of orthogonal matrices are also orthogonal, $\mathbf{F}_n$ has the form

$$\mathbf{F}_n = \prod_{k=0}^{n} \mathbf{P}_k\hat{\mathbf{D}}_k\mathbf{P}_k^{-1}\mathbf{Q}_k = \left(\prod_{k=0}^{n} \|\mathbf{M}_k\|\right)\mathbf{P}_n\left(\prod_{k=0}^{n} \hat{\mathbf{D}}_k\mathbf{O}_k\right), \tag{9}$$

where the orthogonal matrices $\mathbf{O}_k$ ($k = 0, \ldots, n+1$) are defined by

$$\mathbf{O}_0 := \mathbf{P}_0^{-1}\mathbf{Q}_0, \quad \mathbf{O}_k = \mathbf{P}_k^{-1}\mathbf{Q}_k\mathbf{P}_{k-1}, \quad k = 1, \ldots, n, \tag{10}$$

and the product $\prod_{k=0}^{n} \mathbf{N}_k$ is meant in the order $\mathbf{N}_n \cdots \mathbf{N}_0$. It is known from the elementary geometry that in the plane every orthogonal transformation is a rotation or a product of a rotation and a reflection with respect to the x-axis. Thus, if $\mathbf{O}_k$ is not a rotation, then let $\mathbf{O}_k = \mathbf{E}(\vartheta_k)\mathbf{R}$ for some ϑ_k. Since $\mathbf{R}$ is commutable with every diagonal matrices, from (5) we obtain

$$\mathbf{F}_n = \left(\prod_{k=0}^{n} \|\mathbf{M}_k\|\right)\mathbf{R}^m\mathbf{E}(\alpha_n)\left(\prod_{k=0}^{n} \hat{\mathbf{D}}_k\mathbf{E}(\omega_k)\right) \tag{11}$$

for some $m \in \mathbb{N}_0$ ($m \leq n+1$) and some ω_k's, where α_k, ω_k can be calculated from $\mathbf{M}_0, \ldots, \mathbf{M}_k$.

Consider now the difference equation

$$\mathbf{x}_{n+1} = \|\mathbf{M}_n\| \begin{pmatrix} 1 & 0 \\ 0 & d_n \end{pmatrix} \begin{pmatrix} \cos\omega_n & -\sin\omega_n \\ \sin\omega_n & \cos\omega_n \end{pmatrix} \mathbf{x}_n, \qquad (12)$$
$$0 \le d_n \le 1, \qquad n = 0, 1, 2, \dots$$

The equilibrium $(0,0)$ of (1) is stable (asymptotically stable) if and only if the equilibrium $(0,0)$ of (12) is stable (asymptotically stable). Now, we can state the main theorem of this section:

Theorem 1. *Suppose that* $\limsup_{n\to\infty} \prod_{k=0}^{n} \|\mathbf{M}_k\| < \infty$. *If*

$$\sum_{n=0}^{\infty} \min\{1-d_n, 1-d_{n+1}\} \sin^2 \omega_{n+1} = \infty, \qquad (13)$$

then the zero solution of difference equation (12) *is asymptotically stable.*

Proof. Obviously, it is enough to deal with the case $\|\mathbf{M}_k\| = 1$ $(k = 0, 1, \dots)$ and to show that $\left\|\prod_{n=0}^{\infty} \hat{\mathbf{D}}_n \mathbf{E}(\omega_n)\right\| = 0$. Geometrically, the dynamics of (12) is composed of consecutive rotations and contractions along the y-axis. Let us introduce polar coordinates r, φ so that

$$\mathbf{x} := \begin{pmatrix} x \\ y \end{pmatrix}, \quad x = r\sin\varphi, \qquad y = r\cos\varphi.$$

In these coordinates the phase space for system (12) is $r \ge 0$, $-\infty < \varphi < \infty$. Using the notations

$$\tilde{\mathbf{x}}_n = \mathbf{E}(\omega_n)\mathbf{x}_n, \quad \kappa_n := \varphi_{n+1} - (\varphi_n + \omega_n), \quad \Delta r_n := r_{n+1} - r_n, \quad n = 0, 1, \dots$$

we have

$$\sqrt{x_n^2 + y_n^2} = \sqrt{\tilde{x}_n^2 + \tilde{y}_n^2}, \qquad x_{n+1} = \tilde{x}_n, \qquad y_{n+1} = d_n \tilde{y}_n$$
$$\varphi_{n+1} = \varphi_0 + \sum_{i=0}^{n} (\omega_i + \kappa_i), \qquad r_{n+1} = r_0 + \sum_{i=0}^{n} \Delta r_i,$$

and $\Delta r_i \le 0$ because of the contraction. Therefore, the sequence $\{r_n\}_{n=0}^{\infty}$ is monotonously decreasing.

Suppose that the statement of the theorem is not true, i.e., $\bar{r} := \lim_{n\to\infty} r_n > 0$. Then

$$\begin{aligned}
-\Delta r_i = r_i - r_{i+1} &= \sqrt{x_i^2 + y_i^2} - \sqrt{x_{i+1}^2 + y_{i+1}^2} \\
&= \sqrt{\tilde{x}_i^2 + \tilde{y}_i^2} - \sqrt{\tilde{x}_i^2 + d_i^2 \tilde{y}_i^2} = \frac{(1 - d_i^2)\tilde{y}_i^2}{\sqrt{\tilde{x}_i^2 + \tilde{y}_i^2} + \sqrt{\tilde{x}_i^2 + d_i^2 \tilde{y}_i^2}} \\
&\geq \frac{(1 - d_i^2) r_i^2 \cos^2(\varphi_i + \omega_i)}{2r_i} \geq \frac{\bar{r}}{2}(1 - d_i)\cos^2(\varphi_i + \omega_i).
\end{aligned} \tag{14}$$

We want to get the contradiction that the sum of the lower estimating terms in (14) diverges. The problem is that these terms contain φ_i's, which depend on solutions, so they are unknown; we have to get rid of them. Obviously,

$$\begin{aligned}
|\cos(\varphi_i + \omega_i)| &= |\cos\varphi_i \cos\omega_i - \sin\varphi_i \sin\omega_i| \\
&\geq |\sin\varphi_i||\sin\omega_i| - |\cos\varphi_i||\cos\omega_i|.
\end{aligned} \tag{15}$$

For arbitrarily fixed $0 < \gamma < \varepsilon < 1$, define $\mu(\varepsilon, \gamma) := \sqrt{1 - \gamma^2} - \varepsilon\gamma$. Since $\lim_{\varepsilon\to 0, \gamma\to 0} \mu(\varepsilon, \gamma) = 1$, we may assume that $\mu(\varepsilon, \gamma) \geq 1/2$. We distinguish three cases:

a) $\boldsymbol{\gamma|\sin\omega_i| \geq |\cos\varphi_i|}$ **and** $\boldsymbol{|\cos\omega_i| \geq \varepsilon}$**.** Then $|\sin\varphi_i| \geq |\cos\omega_i|$, and from (15) we get

$$|\cos(\varphi_i + \omega_i)| \geq |\sin\omega_i||\cos\omega_i|(1 - \gamma) \geq |\sin\omega_i|(1 - \gamma)\varepsilon. \tag{16}$$

In this case, estimate (14) is continued as

$$-\Delta r_i \geq \frac{\bar{r}}{2}(1 - d_i)\cos^2(\varphi_i + \omega_i) \geq \frac{\bar{r}}{2}(1 - \gamma)^2 \varepsilon^2 (1 - d_i)\sin^2\omega_i. \tag{17}$$

b) $\boldsymbol{\gamma|\sin\omega_i| \geq |\cos\varphi_i|}$ **and** $\boldsymbol{|\cos\omega_i| < \varepsilon}$. Then

$$|\sin\varphi_i| \geq \sqrt{1 - \gamma^2 \sin^2\omega_i} \geq \sqrt{1 - \gamma^2}, \tag{18}$$

and

$$|\cos(\varphi_i + \omega_i)| \geq (\sqrt{1 - \gamma^2} - \varepsilon\gamma)|\sin\omega_i| = \mu(\varepsilon, \gamma)|\sin\omega_i| \geq \frac{1}{2}|\sin\omega_i|.$$

Then

$$-\Delta r_i \geq \frac{\bar{r}}{2}(1 - d_i)\cos^2(\varphi_i + \omega_i) \geq \frac{\bar{r}}{8}(1 - d_i)\sin^2\omega_i. \tag{19}$$

c) $\boldsymbol{\gamma|\sin\omega_i| < |\cos\varphi_i|}$. In this case we can estimate $-\Delta r_{i-1}$ (instead of $-\Delta r_i$) from below by $|\sin\omega_i|$. In fact, using also the inequality

$$\begin{aligned}|\cos\varphi_i| &= \frac{|y_i|}{\sqrt{x_i^2+y_i^2}} = \frac{d_{i-1}|\tilde{y}_{i-1}|}{\sqrt{\tilde{x}_{i-1}^2+d_{i-1}^2\tilde{y}_{i-1}^2}} \\ &\le \frac{|\tilde{y}_{i-1}|}{\sqrt{\tilde{x}_{i-1}^2+\tilde{y}_{i-1}^2}} = |\cos(\varphi_{i-1}+\omega_{i-1})|,\end{aligned} \tag{20}$$

from (14) we obtain

$$\begin{aligned}-\Delta r_{i-1} &\ge \frac{\bar{r}}{2}(1-d_{i-1})\cos^2(\varphi_{i-1}+\omega_{i-1}) \ge \frac{\bar{r}}{2}(1-d_{i-1})\cos^2\varphi_i \\ &\ge \frac{\bar{r}}{2}\gamma^2(1-d_{i-1})\sin^2\omega_i \ge \frac{\bar{r}}{2}\gamma^2\min\{1-d_{i-1},1-d_i\}\sin^2\omega_i.\end{aligned} \tag{21}$$

Setting

$$c := \frac{\bar{r}}{2}\min\{(1-\gamma)^2\varepsilon^2;\frac{1}{4};\gamma^2\} > 0,$$

for every i we have

$$c\min\{1-d_{i-1};1-d_i\}\sin^2\omega_i \le -\Delta r_{i-1}-\Delta r_i = r_{i-1}-r_{i+1}.$$

Summarizing these inequalities we obtain

$$c\sum_{i=1}^{\infty}\min\{1-d_{i-1};1-d_i\}\sin^2\omega_i \le r_0-\bar{r} < \infty,$$

which contradicts assumption (13). $\square$

3 The half-linear equation

In this section we consider the half-linear second order differential equation

$$x''|x'|^{n-1}+q(t)|x|^{n-1}x = 0, \qquad n\in\mathbb{R}^+, \tag{22}$$

which was introduced by Bihari [5] and Elbert [9]. They called it half-linear because its solution set is homogeneous, but it is not additive. This equation is a generalization of the second order linear differential equation

$$x''+q(t)x = 0 \tag{23}$$

describing the motion of a linear oscillator. Following P. Hartman [13, p. 500], we call a non-trivial solution $x_0(t)$ of (22) *small* if

$$\lim_{t\to\infty} x_0(t) = 0. \tag{24}$$

H. Milloux [18] proved, that if q is differentiable, monotonously increasing and tends to infinity as $t \to \infty$, then the linear equation (23) has at least one small solution. He also constructed an equation with such a coefficient q having not small solutions, too. The famous Armellini-Tonelli-Sansone Theorem (see, e.g., [17]) gave a sufficient condition guaranteeing that all solutions of (23) were small. Many papers examined and sharpened the above theorems, even for nonlinear differential equations or difference equations (see, e.g., [15, 17] and the references therein).

F. V. Atkinson and Elbert [4] extended the theorem of H. Milloux to the half-linear differential equation (22). An extension of the A-T-S theorem to (22) was given by Bihari with the following concept. A nondecreasing function $f : [0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} f(t) = \infty$ is called to grow *intermittently* if for every $\varepsilon > 0$ there is a sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ of disjoint intervals such that $a_i \to \infty$ as $i \to \infty$, and

$$\limsup_{i\to\infty} \sum_{k=1}^{i} \frac{b_k - a_k}{b_i} \le \varepsilon, \qquad \sum_{i=1}^{\infty} (f(a_{i+1}) - f(b_i)) < \infty$$

are satisfied. If such a sequence does not exist, then f is called to grow *regularly.*

Theorem B (Bihari [6])**.** *If q is continuously differentiable and it grows to infinity regularly as $t \to \infty$, then all non-trivial solutions of equation* (22) *are small.*

The simplest case of the intermittent growth is when q is a monotonously increasing step function. In this section we will examine this case, i.e., the equation

$$x''|x'|^{n-1} + q_k|x|^{n-1}x = 0 \qquad (t_k \le t < t_{k+1},\ k = 0, 1, \ldots), \tag{25}$$

where

$$t_0 = 0, \qquad \lim_{k\to\infty} t_k = \infty,$$
$$0 < q_0 \le q_1 \le \ldots \le q_k \le q_{k+1} \le \ldots, \qquad \lim_{k\to\infty} q_k = \infty.$$

In [14], the first author of this paper showed that under these conditions equation (25) has a small solution. Elbert [11, 12] proved an A-T-S theorem for the linear ($n = 1$) case of equation (25) as a direct application of his theorem on the asymptotic stability of the trivial solution of (1).

Theorem C (Elbert [11])**.** *Let $n = 1$. If*

$$\sum_{k=0}^{\infty} \min\left\{1 - \frac{q_k}{q_{k+1}}, 1 - \frac{q_{k+1}}{q_{k+2}}\right\} \sin^2(\sqrt{q_{k+1}}(t_{k+2} - t_{k+1})) = \infty, \tag{26}$$

then all non-trivial solutions of equation (25) *are small.*

Our main goal is to extend Theorem C to the case $n > 1$ of half-linear equation (25). To this end, we need the so-called generalized sine and cosine functions introduced by Elbert [9]. Consider the solution $S = S_n(\Phi)$ of the initial value problem

$$\begin{cases} S''|S'|^{n-1} + S|S|^{n-1} = 0 \\ S(0) = 0, \quad S'(0) = 1. \end{cases} \tag{27}$$

Multiplying the differential equation by S' and integrating it over $[0, \Phi]$ we obtain the relation

$$|S'|^{n+1} + |S|^{n+1} = 1 \qquad (-\infty < \Phi < \infty), \tag{28}$$

which can be considered as a generalization of the classical identity $\cos^2 \varphi + \sin^2 \varphi = 1$ (the case $n = 1$). S and S' are periodic functions with period $2\hat{\pi}$, where $\hat{\pi}$ is defined as

$$\hat{\pi} = \frac{2\frac{\pi}{n+1}}{\sin\frac{\pi}{n+1}},$$

which gives back π in the ordinary case $n = 1$ (see [9]). Furthermore, S is odd and S' is even. The generalized tangent function can be introduced as well:

$$T(\Phi) = \frac{S(\Phi)}{S'(\Phi)}.$$

Now we can state our main theorem.

Theorem 2. *Let $n > 1$. If*

$$\sum_{k=0}^{\infty} \min\left\{1 - \frac{q_k}{q_{k+1}}, 1 - \frac{q_{k+1}}{q_{k+2}}\right\} \left| S\left(q_{k+1}^{\frac{1}{n+1}}(t_{k+2} - t_{k+1})\right)\right|^{n+1} = \infty, \tag{29}$$

then all non-trivial solutions of equation (25) *are small.*

Proof. First, using the notation $q(t) := q_k$ $(t_k \le t < t_{k+1},\ k = 0, 1, 2 \dots)$ we introduce a new time variable

$$\tau = \varphi(t) = \int_0^t q(s)^{\frac{1}{n+1}}\, \mathrm{d}s, \qquad \tau_k := \varphi(t_k). \tag{30}$$

Let $x(t) = x(\varphi^{-1}(\tau)) =: y(\tau)$, where φ^{-1} is the inverse function of φ. Then

$$x'(t) = \dot{y}(\tau) q^{\frac{1}{n+1}}(t), \quad x''(t) = \ddot{y}(\tau) q^{\frac{2}{n+1}}(t) \qquad (t \ne t_k,\ k = 0, 1, 2, \dots),$$

where $(\cdot)^{\cdot} = \mathrm{d}(\cdot)/\mathrm{d}\tau$. Thus, equation (25) is transformed into the form

$$\ddot{y}(\tau)|\dot{y}(\tau)|^{n-1} + |y(\tau)|^{n-1} y(\tau) = 0, \qquad (\tau \ne \tau_k\ k = 0, 1, \dots). \tag{31}$$

Since any solution x of equation (25) has to be continuously differentiable on $(0, \infty)$, $x'(t_{k+1} - 0) = x'(t_{k+1} + 0) = x'(t_{k+1})$ must hold for every $k \in \mathbb{N}$, i.e.,

$$\dot{y}(\tau_{k+1}) = \dot{y}(\tau_{k+1} + 0) = \left(\frac{q_k}{q_{k+1}}\right)^{\frac{1}{n+1}} \dot{y}(\tau_{k+1} - 0),$$

where $f(t-0)$ and $f(t+0)$ denotes the left-hand side and the right-hand side limit of a function f at t, respectively. We obtain that (25) is equivalent to the following differential equation with impulses:

$$\begin{cases} \ddot{y}(\tau)|\dot{y}(\tau)|^{n-1} + |y(\tau)|^{n-1} y(\tau) = 0, & \tau \ne \tau_k \\ \dot{y}(\tau_{k+1}) = \left(\frac{q_k}{q_{k+1}}\right)^{\dfrac{1}{n+1}} \dot{y}(\tau_{k+1} - 0), & k = 0, 1, 2, \dots \end{cases} \tag{32}$$

Let us introduce the generalized polar coordinates $\dot{y} = \rho S'(\Phi)$, $y = \rho S(\Phi)$, where

$$\rho = (|\dot{y}|^{n+1} + |y|^{n+1})^{\frac{1}{n+1}}, \qquad T(\Phi) = \frac{y}{\dot{y}}, \quad -\infty < \Phi < \infty.$$

This is the so-called generalized Prüfer transformation. With the aid of these variables we can rewrite equation (31) into

$$\dot{\Phi} = 1, \qquad \dot{\rho} = 0, \qquad (\tau_k \le \tau < \tau_{k+1},\ k = 0, 1, \dots). \tag{33}$$

So the dynamics of system (32) on the Minkowski plane [19] $(\dot{y}, y)$ is the following. It turns any point $(\dot{y}_0, y_0)$ around the origin on the Minkowski circle with radius $\rho_0 := (|\dot{y}_0|^{n+1} + |y_0|^{n+1})^{\frac{1}{n+1}}$ on $[\tau_0, \tau_1)$, and at τ_1 the point $(\dot{y}(\tau_1 - 0), y(\tau_1 - 0))$ jumps to the point

$$(\dot{y}(\tau_1), y(\tau_1)) := \left(\left(\frac{q_0}{q_1} \right)^{\frac{1}{n+1}} \dot{y}(\tau_1 - 0), y(\tau_1 - 0) \right).$$

This process is repeated consecutively for $[\tau_1, \tau_2)$, $[\tau_2, \tau_3), \ldots$. Define

$$\begin{gathered} \rho_k := \left(|\dot{y}(\tau_k)|^{n+1} + |y(\tau_k)|^{n+1}\right)^{\frac{1}{n+1}}, \quad \Phi_k := \Phi(\tau_k), \quad \Omega_k := \tau_{k+1} - \tau_k, \\ \Delta\rho_k := \rho_{k+1} - \rho_k, \qquad \kappa_k := \Phi_{k+1} - (\Phi_k + \Omega_k), \qquad k = 0, 1, \ldots \end{gathered}$$

Obviously,

$$\Phi_{k+1} = \Phi_0 + \sum_{i=0}^{k} (\Omega_i + \kappa_i), \qquad \rho_{k+1} = \rho_0 + \sum_{i=0}^{k} \Delta\rho_i, \quad k = 0, 1 \ldots$$

Since $\Delta\rho_i \leq 0$, the sequence $\{\rho_k\}_{k=0}^{\infty}$ is monotonously decreasing, therefore it has a limit $\bar{\rho} := \lim_{k\to\infty} \rho_k$. If the statement of the theorem is not true, then there exists a solution (ρ, Φ) such that $\bar{\rho} > 0$. Let us consider this solution and estimate $-\Delta\rho_i$:

$$\begin{aligned} -\Delta\rho_i &= \rho_i - \rho_{i+1} \\ &= (|\dot{y}(\tau_i)|^{n+1} + |y(\tau_i)|^{n+1})^{\frac{1}{n+1}} - (|\dot{y}(\tau_{i+1})|^{n+1} + |y(\tau_{i+1})|^{n+1})^{\frac{1}{n+1}} \\ &= (|\dot{y}(\tau_{i+1} - 0)|^{n+1} + |y(\tau_{i+1} - 0)|^{n+1})^{\frac{1}{n+1}} \\ &\quad - (|\dot{y}(\tau_{i+1})|^{n+1} + |y(\tau_{i+1})|^{n+1})^{\frac{1}{n+1}} \\ &= (|\dot{y}(\tau_{i+1} - 0)|^{n+1} + |y(\tau_{i+1} - 0)|^{n+1})^{\frac{1}{n+1}} \\ &\quad - \left(\frac{q_i}{q_{i+1}} |\dot{y}(\tau_{i+1} - 0)|^{n+1} + |y(\tau_{i+1} - 0)|^{n+1} \right)^{\frac{1}{n+1}} \\ &= \frac{1}{n+1} \left(\rho_{i+1}^{n+1} + \eta_i \left(\rho_i^{n+1} - \rho_{i+1}^{n+1} \right) \right)^{-\frac{n}{n+1}} \\ &\quad \times \left(1 - \frac{q_i}{q_{i+1}} \right) |\dot{y}(\tau_{i+1} - 0)|^{n+1} \\ &\geq \frac{1}{n+1} \left((\bar{\rho})^{n+1} \right)^{-\frac{n}{n+1}} \left(1 - \frac{q_i}{q_{i+1}} \right) \rho_i^{n+1} |S'(\Phi_i + \Omega_i)|^{n+1} \\ &\geq \frac{\bar{\rho}}{n+1} \left(1 - \frac{q_i}{q_{i+1}} \right) |S'(\Phi_i + \Omega_i)|^{n+1} \end{aligned} \tag{34}$$

with some $\eta_i \in (0,1)$ for all $i \in \mathbb{N}_0$. Now we need to estimate $|S'(\phi_i + \Omega_i)|$ from below by either $|S(\Omega_i)|$ or $|S(\Omega_{i+1})|$, similarly to the proof of Theorem 1, where we used the addititonal formulae for the cosine function. However, to our best knowledge, the problem of finding exact addition formulae for S and S' is not completely solved, although there are some papers about this topic (see, e.g., [1], [2]). Therefore, to complete the proof we need a new method different from one we used in the proof of Theorem 1 after formula (14).

Functions $|S'(\Phi + \Omega)|$ and $|S(\Omega)|$ are $\hat{\pi}$-periodic with respect to both variables Φ, Ω, hence we may restrict ourselves to the quadrant $[-\hat{\pi}/2, \hat{\pi}/2] \times [-\hat{\pi}/2, \hat{\pi}/2]$ on the (Φ, Ω) plane. Thanks to the symmetry properties of S and S', it is enough to make the estimate on $Q := [0, \hat{\pi}/2] \times [0, \hat{\pi}/2]$.

At first, let us handle the set

$$Q_\varepsilon := \{(\Phi, \Omega) \in Q : |S'(\Phi)| < \varepsilon\},$$

where $\varepsilon > 0$ is small enough. The complementer set of Q_ε with respect to Q will be treated in another way. The same way will be used also for the complementer set of

$$Q^\gamma := \{(\Phi, \Omega) \in Q : |S'(\Phi)| \leq \gamma |S(\Omega)|\} \quad (0 < \gamma < 1),$$

so now we consider the set $Q_\varepsilon^\gamma := Q_\varepsilon \cap Q^\gamma$ (see the figure).

A part of the boundary of this set is a piece of the curve defined by the equation

$$\Gamma : |S'(\Phi)| = \gamma |S(\Omega)|.$$

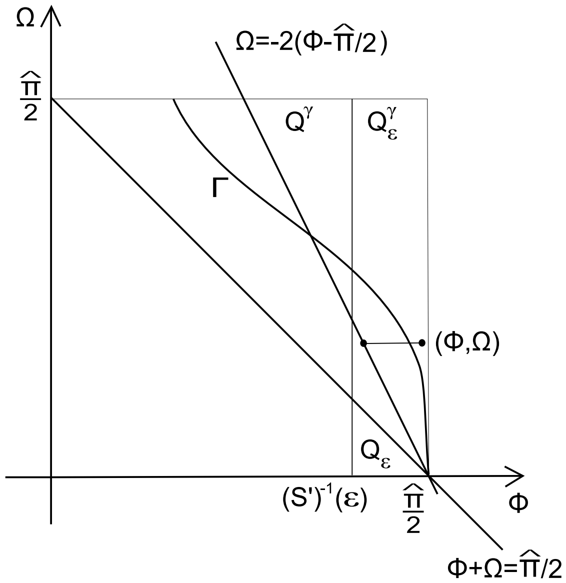
We show that the tangent to Γ at $(\hat{\pi}/2, 0)$ is the line $\Phi = \hat{\pi}/2$, i.e.,

$$\lim_{\Phi \to \frac{\pi}{2} - 0} f'(\Phi) = -\infty; \quad f(\Phi) := S^{-1}\left(\frac{1}{\gamma} S'(\Phi)\right), \tag{35}$$

provided $n > 1$. The statement of the theorem for the linear case $n = 1$ was proved in Theorem 1, so proving (35) we can restrict ourselves to the case $n > 1$.

It is easy to see that

$$(S^{-1})'(W) = \frac{1}{(1 - W^{n+1})^{\frac{1}{n+1}}} \quad (0 \leq W \leq 1).$$



Besides, by equation (27) we have

$$S''(\Phi) = -|S'(\Phi)|^{-n+1}|S(\Phi)|^{n-1}S(\Phi). \tag{36}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}f(\Phi) = f'(\Phi) = \frac{-\frac{1}{\gamma}(S'(\Phi))^{-n+1}S^n(\Phi)}{\left(1 - \frac{1}{\gamma^{n+1}}(S'(\Phi))^{n+1}\right)^{\frac{1}{n+1}}},$$

consequently, (35) holds, independently of γ. (35) implies the existence of a

$\delta > 0$ such that

$$f'(\Phi) < -2 \quad \left((S')^{-1}(\varepsilon) < \frac{\hat{\pi}}{2} - \delta < \Phi < \frac{\hat{\pi}}{2}\right),$$

whence we get

$$f(\Phi) \geq -2\left(\Phi - \frac{\hat{\pi}}{2}\right),$$

which means that Γ is located on the right-hand side of the line $\Omega = -2(\Phi - \hat{\pi}/2)$ near the point $(\hat{\pi}/2, 0)$ (see the figure). To estimate $|S'(\Phi_i + \Omega_i)|$ from below by $|S(\Omega_i)|$ in (34) we have to estimate the quotient $|S'(\Phi + \Omega)|\,/|S(\Omega)|$ from below. In Q_ε^γ we decrease this quotient exchanging point (Φ, Ω) for the horizontally corresponding point $(\hat{\pi}/2 - \Omega/2, \Omega)$ of the line $\Phi = \hat{\pi}/2 - \Omega/2$ (see the figure again). Therefore, by the L'Hospital Rule and (36) we get

$$\begin{aligned}
&\lim_{\Phi \to \frac{\hat{\pi}}{2}-0,\, \Omega \to 0+0,\, (\Phi,\Omega) \in Q_\varepsilon^\gamma} \frac{|S'(\Phi+\Omega)|}{|S(\Omega)|} \geq \lim_{\Omega \to 0+0} \frac{-S'\left(\left(\frac{\hat{\pi}}{2} - \frac{1}{2}\Omega\right) + \Omega\right)}{S(\Omega)} \\
&= \lim_{\Omega \to 0+0} \frac{-S'\left(\frac{\hat{\pi}}{2} + \frac{1}{2}\Omega\right)}{S(\Omega)} = \lim_{\Omega \to 0+0} \frac{-S''\left(\frac{\hat{\pi}}{2} + \frac{1}{2}\Omega\right)\frac{1}{2}}{S'(\Omega)} \\
&= \lim_{\Omega \to 0+0} \frac{\left|S'\left(\frac{\hat{\pi}}{2} + \frac{\Omega}{2}\right)\right|^{-n+1} \left|S\left(\frac{\hat{\pi}}{2} + \frac{\Omega}{2}\right)\right|^{n-1} S\left(\frac{\hat{\pi}}{2} + \frac{\Omega}{2}\right)}{2S'(\Omega)} = \infty.
\end{aligned}$$

This means that there exists a $\kappa > 0$ such that

$$|S'(\Phi + \Omega)| \geq \kappa |S(\Omega)| \qquad ((\Phi, \Omega) \in Q_\varepsilon^\gamma). \tag{37}$$

Now we are ready to complete estimate (34). We distinguish three cases:

A) $(\Phi_i, \Omega_i) \in Q_\varepsilon^\gamma$. Then by (34) and (37) we have

$$-\Delta\rho_i \geq \frac{\overline{\rho}}{n+1}\left(1 - \frac{q_i}{q_{i+1}}\right)\kappa^{n+1}|S(\Omega_i)|^{n+1}. \tag{38}$$

In the remaining cases we estimate $-\Delta\rho_{i-1}$. By the analogue of (20) it is always true that

$$\begin{aligned}
-\Delta\rho_{i-1} &\geq \frac{\overline{\rho}}{n+1}\left(1 - \frac{q_{i-1}}{q_i}\right)|S'(\Phi_{i-1} + \Omega_{i-1})|^{n+1} \\
&\geq \frac{\overline{\rho}}{n+1}\left(1 - \frac{q_{i-1}}{q_i}\right)|S'(\Phi_i)|^{n+1}.
\end{aligned}$$

B) $(\boldsymbol{\Phi_i, \Omega_i}) \in \boldsymbol{Q_\varepsilon \backslash Q_\varepsilon^\gamma}$. Then $|S'(\Phi_i)| \geq \gamma |S(\Omega_i)|$, and

$$-\Delta\rho_{i-1} \geq \gamma^{n+1} \frac{\overline{\rho}}{n+1} \left(1 - \frac{q_{i-1}}{q_i}\right) |S(\Omega_i)|^{n+1}. \qquad (39)$$

C) $(\boldsymbol{\Phi_i, \Omega_i}) \in \boldsymbol{Q \backslash Q_\varepsilon}$. Then $|S'(\Phi_i)| \geq \varepsilon |S(\Omega_i)|$ and

$$-\Delta\rho_{i-1} \geq \varepsilon^{n+1} \frac{\overline{\rho}}{n+1} \left(1 - \frac{q_{i-1}}{q_i}\right) |S(\Omega_i)|^{n+1}. \qquad (40)$$

Setting

$$C := \frac{\overline{\rho}}{n+1} \min\{\kappa^{n+1}; \gamma^{n+1}; \varepsilon^{n+1}\} > 0,$$

and taking into account (38), (39), (40), for every i we have

$$C \min\left\{1 - \frac{q_{i-1}}{q_i}; 1 - \frac{q_i}{q_{i+1}}\right\} |S(\Omega_i)|^{n+1} \leq \Delta\rho_{i-1} - \Delta\rho_i = \rho_{i-1} - \rho_{i+1}.$$

Summarizing these inequalities we obtain

$$C \sum_{n=1}^{\infty} \min\left\{1 - \frac{q_{i-1}}{q_i}; 1 - \frac{q_i}{q_{i+1}}\right\} |S(\Omega_i)|^{n+1} \leq \rho_0 - \overline{\rho} < \infty,$$

which contradicts the assumption of the theorem. $\square$

Theorem 2 extends Elbert's Theorem C to half-linear equations provided $n > 1$. It would bee interesting to find an extension to the case $n < 1$, too.

References

[1] I. Adamaszek, On generalized sine and cosine functions, *Demonstratio Math.* **28**(1995), 263–270.

[2] I. Adamaszek-Fidytek, On generalized sine and cosine functions, II. *Demonstratio Math.* **37**(2004), 571–578.

[3] R. P. Agarwal, *Difference Equations and Inequalities*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 115, Marcel Dekker, Inc., New York, 1992.

[4] F. V. Atkinson, Á. Elbert, An extension of Milloux's theorem to half-linear differential equations, Proc. 6'th Colloq. Qual. Theory Differ. Equ., No. 8, *Electron. J. Qual. Theory Differ. Equ., Szeged*, 2000.

[5] I. Bihari, Ausdehnung der Sturmschen Oscillations und Vergleichssätze auf die Lösungen gewisser nichtlinearen Differentialgleichungen zweiter Ordnung, *Publ. Math. Inst. Hungar. Acad. Sci.*, **2**(1957), 154–165.

[6] I. Bihari, Asymptotic result concerning equation $x''|x'|^{n-1} + a(t)x^{\overset{*}{n}}$. Extension of a theorem by Armellini-Tonelli-Sansone, *Studia Sci. Math. Hungar.* **19**(1984), 151–157.

[7] O. Došlý, Half-linear differential equations, *Handbook of Differential Equations*, 161–357, Elsevier/North-Holland, Amsterdam, 2004.

[8] O. Došlý, P. Řehák, *Half-linear Differential Equations*, Elsevier/North-Holland, Amsterdam, 2005.

[9] Á. Elbert, A half-linear second order differential equation, *Qualitative Theory of Differential Equations, Vol. I, II (Szeged, 1979)*, 153–180, *Colloq. Math. Soc. János Bolyai*, **30**, North-Holland, Amsterdam-New York, 1981.

[10] Á. Elbert, Stability of some difference equations, *Advances in Difference Equations: Proceedings of the Second International Conference on Difference Equations and Applications, Veszprém, Hungary, 7-11 August 1995*, Gordon and Breach Science Publishers, 1997, 155–178.

[11] Á. Elbert, On asymptotic stability of some Sturm-Liouville differential equations, *General Seminar in Mathematics*, University of Patras, **22-23**(1996/97), 57–66.

[12] Á. Elbert, On damping of linear oscillators, *Studia Sci. Math. Hungar.* **38**(2001), 191–208.

[13] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.

[14] L. Hatvani, On stability properties of solutions of second order differential equations, Proc. 6'th Colloq. Qual. Theory Differ. Equ., No. 11, *Electron. J. Qual. Theory Differ. Equ., Szeged*, 2000.

[15] L. Hatvani, The growth condition guaranteeing small solutions for a linear oscillator with an increasing elasticity coefficient, *Georgian Math. J.* **14**(2007), no. 2, 269–278.

[16] N. Jacobson, *Lectures in Abstract Algebra, II. Linear Algebra*, Springer Verlag (New York - Heidelberg - Berlin), 1953.

[17] J. W. Macki, Regular growth and zero tending solutions, *Ordinary Differential Equations and Operators (Dundee, 1982)*, Lecture Notes in Math., vol. 1032, 358–399.

[18] H. Milloux, Sur l'equation differentielle $x''+A(t)x = 0$, *Prace Mat.-Fiz.*, **41**(1934), 39–54.

[19] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer, Berlin, 1959.

(Received April 2, 2011)